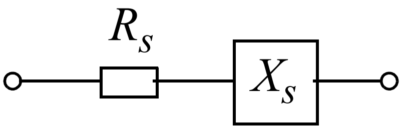
CHAPTER 2

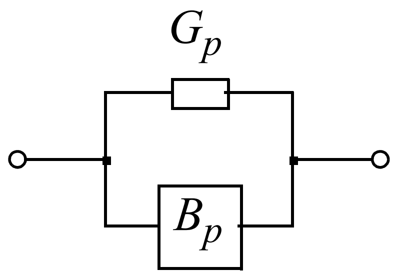
RESONANT CIRCUITS

SERIES AND PARALLEL CIRCUITS

It is useful to remember that an impedance Z (admittance Y) can always be represented by $Z = R_s + jX_s$ ($Y = G_p + jB_p$) where R_s is the resistance and X_s the reactance (G_p is the conductance and B_p the susceptance). This representation is equivalent to a series (parallel) connection of a resistance R_s (conductance G_p) and a reactance X_s (susceptance B_p) (cf Fig. 2-1).



a) Series circuit.



b) Parallel circuit.

Fig 2-1: *Series and parallel circuits.*

The impedance and the admittance being inverses of each other, they are related by:

$$Z \equiv R_s + jX_s = \frac{1}{Y} = \frac{1}{G_p + jB_p} = \frac{G_p - jB_p}{G_p^2 + B_p^2}$$
$$Y \equiv G_p + jB_p = \frac{1}{Z} = \frac{1}{R_s + jX_s} = \frac{R_s - jX_s}{R_s^2 + X_s^2} \quad (2.1)$$

where:

$$R_s = \frac{G_p}{G_p^2 + B_p^2} \qquad X_s = \frac{-B_p}{G_p^2 + B_p^2}$$
$$G_p = \frac{R_s}{R_s^2 + X_s^2} \qquad B_p = \frac{-X_s}{R_s^2 + X_s^2} \quad (2.2)$$

QUALITY FACTOR

Inductors introduce losses represented by a series resistance. The ratio between the energy stored in the inductance and the energy dissipated in the resistance is defined as the quality factor of the inductor. Likewise, the quality factor of a capacitor is the ratio between the energy stored in the capacitance and the energy dissipated in the series resistance which represents the losses in the dielectric.
In general, the quality factor of a series circuit and a parallel circuit is defined by:

$$\boxed{Q \equiv \frac{|X_s|}{R_s} = \frac{|B_p|}{G_p} = \frac{R_p}{|X_p|}} \tag{2.3}$$

In the cases of inductive and capacitive circuits, we have respectively:

$$Q = \frac{\omega L_s}{R_s} = \frac{R_p}{\omega L_p} \qquad Q = \frac{1}{\omega R_s C_s} = \omega R_p C_p \tag{2.4}$$

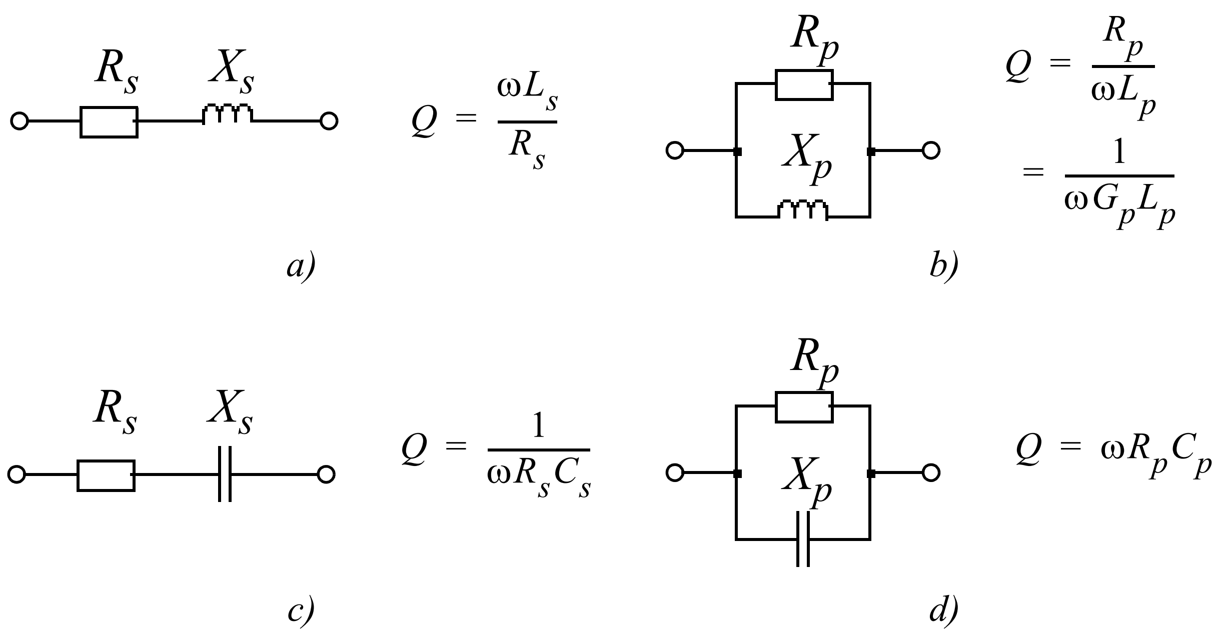


Fig 2-2: *Definition of quality factor.*

SERIES / PARALLEL TRANSFORMATION

It is often useful in HF to be able to convert a series circuit to its parallel equivalent or vice versa, as indicated in Fig. 2-3. This technique is used for example to synthesize an impedance matching network between two given impedances.

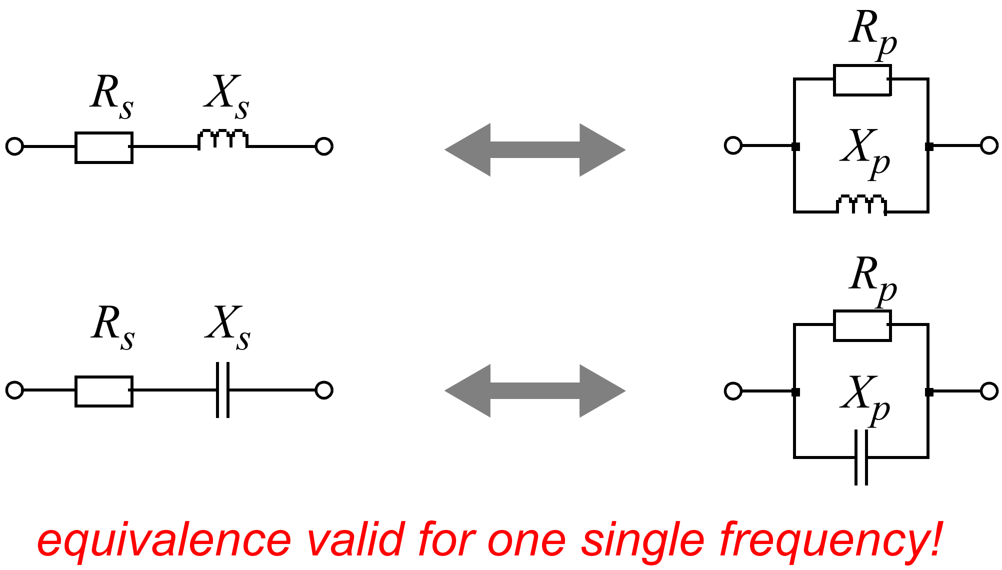


Fig 2-3: *Series / parallel conversion.*

When a series circuit is converted to its parallel equivalent (or vice versa), it is useful to express Eqn. 2.2 as a function of the quality factor of the circuit:

$$R_s = \frac{R_p}{1+Q^2} \qquad X_s = \frac{X_p}{1+1/Q^2} \tag{2.5}$$

$$R_p = R_s(1+Q^2) \qquad X_p = X_s(1+1/Q^2)$$

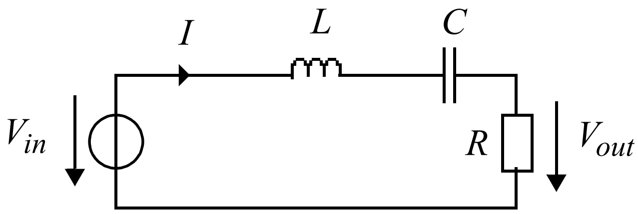
where $R_p \equiv 1/G_p$ and $X_p \equiv -1/B_p$. Note that the quality factor of the series circuit is identical to that of the parallel circuit. It is given by Eqn. 2.3 or 2.4.

In addition, it is important to notice that the reactances of series and parallel circuits depend on the frequency and therefore that this circuit conversion is only valid *for one single frequency*.

SERIES RESONANT CIRCUITS

The impedance of the series resonant circuit shown in Fig. 2-4 is given by:

$$Z \equiv \frac{V_{in}}{I} = R + j(X_L + X_C) \tag{2.6}$$



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} = \frac{1}{R} \cdot \sqrt{\frac{L}{C}}$$

Fig 2-4: *Series resonant circuit.*

This impedance is minimum (and therefore the current is maximum for a constant amplitude of applied voltage), when:

$$-X_L = X_C \qquad or \qquad \omega_0 L = \frac{1}{\omega_0 C} \tag{2.7}$$

This equality is only true at one frequency, called the resonant frequency f_0 (or ω_0 in radians):

$$\boxed{\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}} \tag{2.8}$$

At this frequency, the reactance of the inductor is compensated by the reactance of the capacitor, and therefore the output voltage V_{out} is equal to the input voltage V_{in} . The transfer function for the voltage is given by:

$$A_v(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} = \frac{sRC}{s^2 LC + sRC + 1} = \frac{\frac{s}{\omega_0 Q}}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0 Q} + 1} \tag{2.9}$$

POLES

This circuit has two poles given by:

$$p_{1,2} = \begin{cases} -\dfrac{\omega_0}{2Q} \pm \omega_0 \cdot \sqrt{1/(2Q)^2 - 1} & \textit{for: } Q < 1/2 \quad \textit{(real poles)} \\ -\dfrac{\omega_0}{2Q} \pm j\omega_0 \cdot \sqrt{1 - 1/(2Q)^2} & \textit{for: } Q > 1/2 \quad \textit{(complex poles)} \end{cases} \tag{2.10}$$

The poles corresponding to Eqn. 2.10 are presented in Fig. 2-5. The poles are real for $Q < 1/2$ ($R > 2\sqrt{L/C}$), confounded for $Q = 1/2$ ($R = 2\sqrt{L/C}$) and complex for $Q > 1/2$ ($R < 2\sqrt{L/C}$).

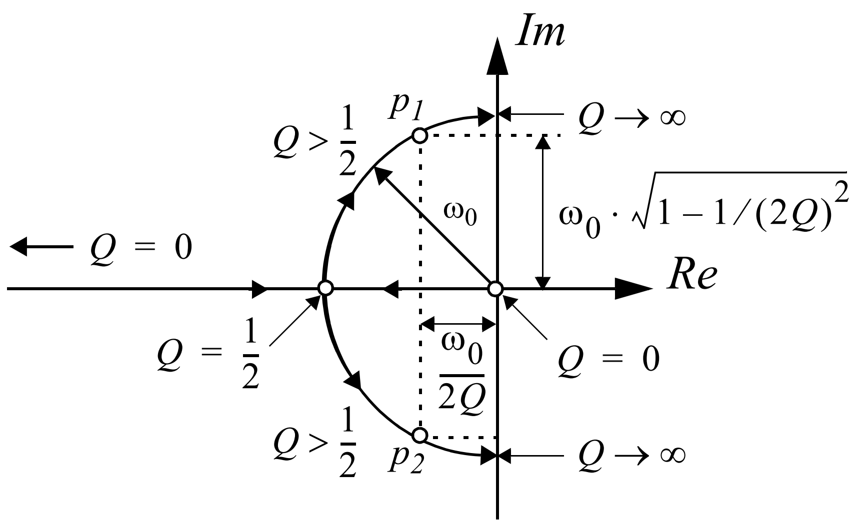


Fig 2-5: *Poles of the transfer function.*

The harmonic response is obtained by replacing s by $j\omega$ in Eqn. 2.9:

$$A_v(j\omega) = \frac{j\dfrac{\omega}{\omega_0 Q}}{1 - \left(\dfrac{\omega}{\omega_0}\right)^2 + j\dfrac{\omega}{\omega_0 Q}} = \frac{1}{1 + jQ\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)} = \frac{1}{1 + jQx} \tag{2.11}$$

where x is the misalignment. For $\Delta\omega \equiv (\omega - \omega_0) << \omega_0$ the misalignment is approximately equal to $2\Delta\omega / \omega_0$ and therefore proportional to the distance of the frequency from the resonance.

BANDWIDTH

The power dissipated in the resistor is given by:

$$P_R = \frac{V_{out}^2}{R} = |A_v(j\omega)|^2 \frac{V_{in}^2}{R} = |A_v(j\omega)|^2 P_{max} \qquad (2.12)$$

It is maximum when $|A_v(j\omega)|^2 = 1$, meaning at the resonance. We therefore define the frequencies at –3 dB as those for which the power dissipated in the resistor is equal to half of the maximum power P_{max}, which corresponds to the frequencies ω_1 and ω_2 for which $|A_v(j\omega)| = 1/\sqrt{2}$ or where $x = \pm 1/Q$:

$$x_1 \equiv \frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1} = -\frac{1}{Q} \qquad x_2 \equiv \frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2} = +\frac{1}{Q} \qquad (2.13)$$

Due to the geometric symmetry of $|A_v(j\omega)|$ around ω_0, the frequencies at –3 dB, ω_1 and ω_2, are such that:

$$\omega_1 \omega_2 = \omega_0^2 \qquad (2.14)$$

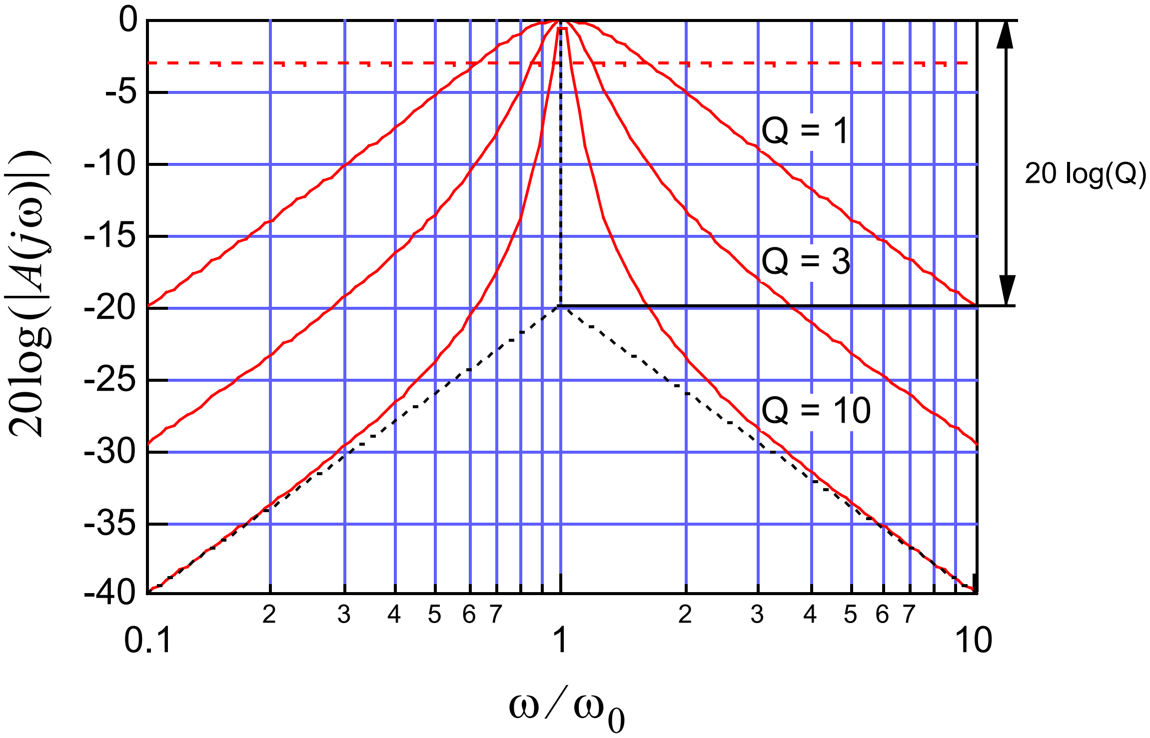
ω_1 and ω_2 can be calculated from:

$$\frac{\omega_0^2}{\omega_2} = \omega_1 \qquad et: \qquad \omega_2 - \frac{\omega_0^2}{\omega_2} = \omega_2 - \omega_1 = \frac{\omega_0}{Q} \qquad (2.15)$$

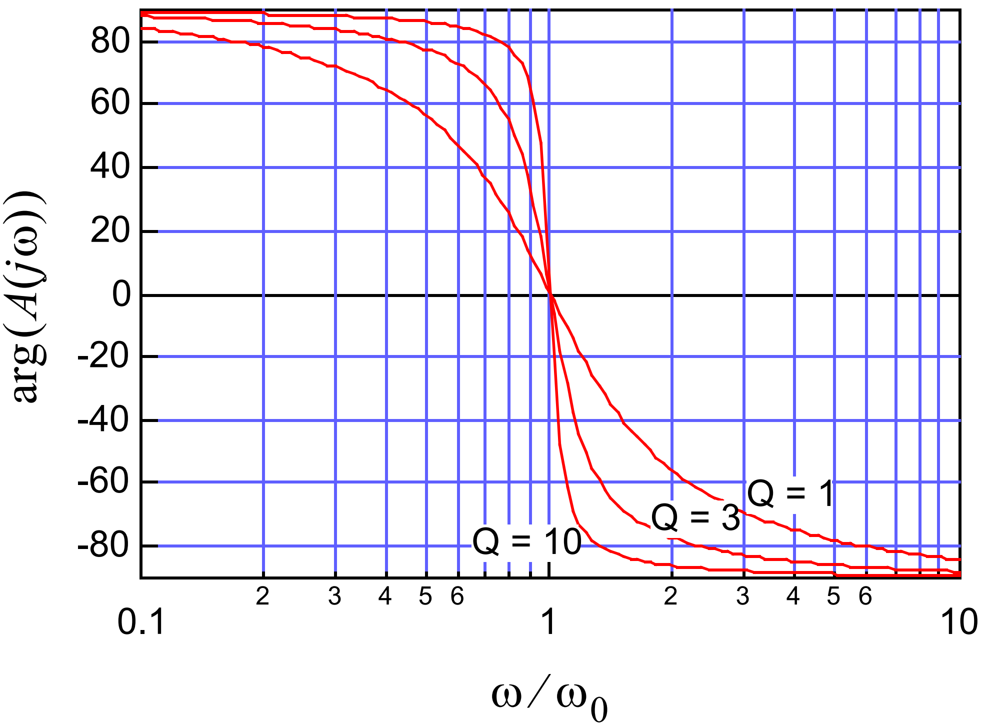
The difference between the frequencies at –3 dB is defined as the bandwidth at –3 dB:

$$B_{-3dB} \equiv \omega_2 - \omega_1 = \frac{\omega_0}{Q} = \omega_0^2 \cdot RC = \frac{R}{L} \qquad (2.16)$$

HARMONIC RESPONSE



a) Amplitude.



b) Phase.

Fig 2-6: *Transfer function* $A(j\omega)$.

FILTERING OF HARMONICS

The attenuation of a sinusoidal signal which is a harmonic frequency of the resonant frequency $\omega = n \cdot \omega_0$ can be calculated from:

$$\left|A(n\omega_0)\right| = \frac{1}{\sqrt{1 + Q^2(n - 1/n)^2}} \underset{Q>>1}{\cong} \frac{n}{Q(n^2 - 1)} \underset{n>>1}{\cong} \frac{1}{Qn} \qquad (2.17)$$

The attenuation of harmonics is better when the quality factor is higher. This criterion can help when choosing the quality factor of a resonant circuit.

VOLTAGE AT THE TERMINALS OF THE CAPACITOR AT RESONANCE

One interesting property of the series resonant circuit is that the voltage at the terminals of the capacitor can become much higher than the applied voltage. The transfer function between the input voltage and the voltage at the capacitor terminals is given by:

$$A_C(j\omega) \equiv \frac{V_C}{V_{in}} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{1}{Q}} \qquad (2.18)$$

At resonance, the gain is equal to the quality factor:

$$\left|A_C(\omega = \omega_0)\right| = Q \qquad (2.19)$$

At resonance, the voltage at the capacitor terminals is thus equal to Q times the input voltage. Since the reactances of the inductor and the capacitor are equal at resonance, the voltage at the inductor terminals will also be Q times larger than the input voltage.

EFFECT OF THE SOURCE RESISTANCE

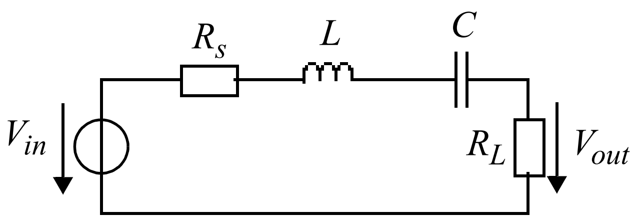


Fig 2-7: *Series resonant circuit with source resistance.*

In the case in which the source resistance R_s is not negligible with respect to the load resistance R_L, the transfer function of the circuit in Fig. 2-7 is given by:

$$A_v(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} = A_0 \cdot \frac{\frac{s}{\omega_0 Q}}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0 Q} + 1} \tag{2.20}$$

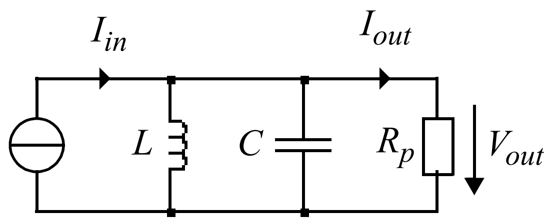
The transfer function given by Eqn. 2.9 is multiplied by the attenuation factor $A_0 = R_L / (R_s + R_L)$ introduced by the resistive divider operating at resonance. The quality factor is degraded by the presence of the source resistance:

$$Q = \frac{\omega_0 L}{R_s + R_L} \tag{2.21}$$

The addition of a series resistance decreases the quality factor without changing the resonant frequency which remains equal to $\omega_0 = 1/\sqrt{LC}$.

PARALLEL RESONANT CIRCUIT

The parallel resonant circuit shown in Fig. 2-8 is the dual of the series resonant circuit of Fig. 2-4. It has, therefore, all the properties stated for the series resonant circuit. In this case, the admittance has a minimum when the susceptance of the inductor is equal to that of the capacitor. Again, this equality is only true at the resonant frequency given by Eqn. 2.8. The gain in current $A_i(s) \equiv I_{out}/I_{in}$ is identical to Eqn. 2.9.



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{R_p}{\omega_0 L} = \omega_0 R_p C = R_p \cdot \sqrt{\frac{C}{L}}$$

Fig 2-8: *Resonant parallel circuit.*

The quality factor is simply given by:

$$Q = \frac{R_p}{\omega_0 L} = \omega_0 R_p C = R_p \cdot \sqrt{\frac{C}{L}} \tag{2.22}$$

In contrast to the series resonant circuit, the quality factor of the parallel resonant circuit is proportional to the parallel resistance. The higher this resistance, the higher the quality factor. The impedance of the parallel resonant circuit is simply:

$$Z(s) \equiv \frac{V_{out}}{I_{in}} = R_p \cdot A_i(s) = R_p \cdot \frac{s\frac{L}{R_p}}{s^2 LC + s\frac{L}{R_p} + 1} = R_p \cdot \frac{\frac{s}{\omega_0 Q}}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0 Q} + 1} \tag{2.23}$$

As with the voltage at the terminals of the capacitor and the inductor of the series resonant circuit, the currents through those same elements of the parallel resonant circuit are Q times higher than the input current I_{in} at resonance.

PARALLEL RESONANT CIRCUIT WITH LOSSES

The impedance of the circuit in Fig. 2-9 is given by:

$$Z(s) = r_s \cdot \frac{s\frac{L}{r_s} + 1}{s^2LC + sr_sC + 1} \cong r_s \cdot \frac{s\frac{L}{r_s}}{s^2LC + sr_sC + 1} \qquad for:\ \omega >> \frac{r_s}{L} \qquad (2.24)$$

For $\omega >> r_s/L$, the zero can be considered to be practically at the origin and Eqn. 2.24 is identical to Eqn. 2.23 of the parallel resonant circuit, as long as $L/R_p = r_sC$ or:

$$R_p = \frac{L}{r_sC} = \frac{(\omega_0 L)^2}{r_s} = r_s \cdot Q_L^2 \qquad (2.25)$$

$$Q_L = \frac{\omega_0 L}{r_s} \qquad Q_{circuit} = \frac{R_p}{\omega_0 L} = \frac{\omega_0 L}{r_s} = Q_L$$

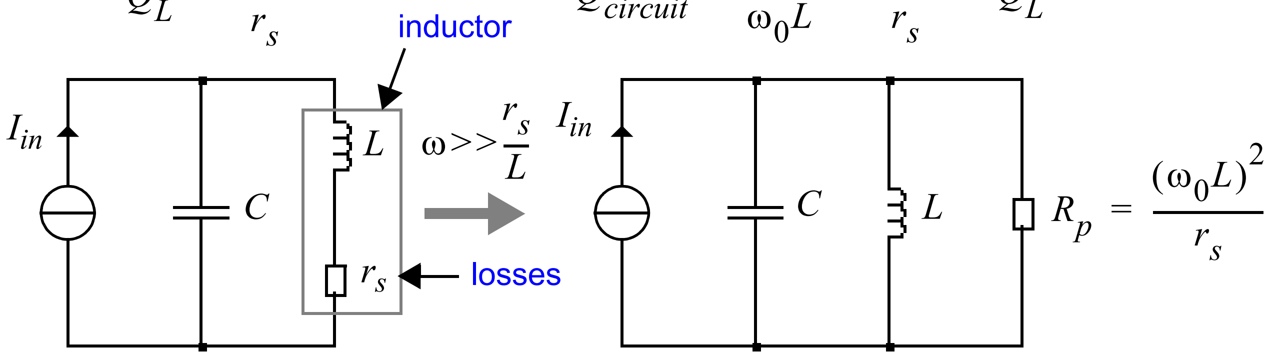
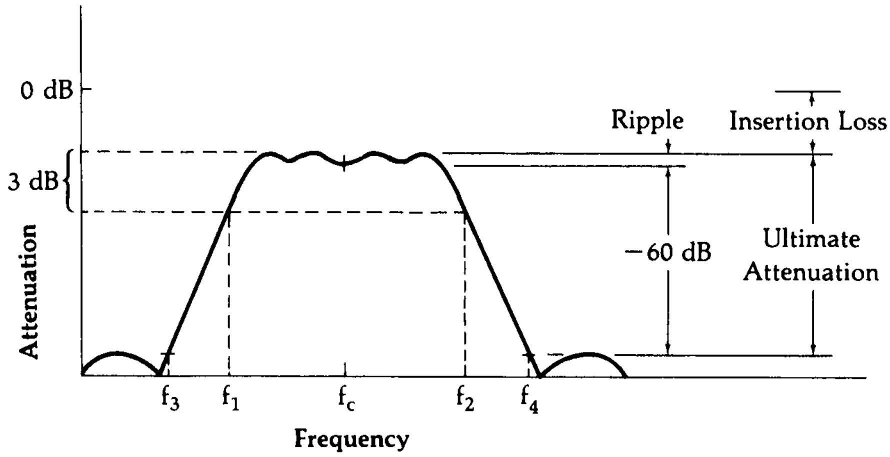


Fig 2-9: *Approximation of the resonant circuit with losses.*

Therefore, for $\omega >> r_s/L$, the resonant circuit with a resistance r_s in series with the inductor can be replaced by a parallel resonant circuit with a parallel resistance R_p given by Eqn. 2.25. This approximation is almost always valid because one does not choose an inductor with a mediocre quality factor to make a circuit with a high quality factor. It is interesting to note that the quality factor of the parallel equivalent circuit defined as $Q = R_p/(\omega_0 L)$ is identical to the quality factor of the inductor $Q_L = (\omega_0 L)/r_s$. This quality factor is often called the unloaded quality factor (without load). The addition of a parallel load resistance will reduce the quality factor of the circuit.

COUPLING PARALLEL RESONANT CIRCUITS

For a given resonant frequency, the quality factor sets the bandwidth, but the sharpness of the filter is uniquely determined by the filter's order, that is to say by the number of reactive components. To obtain a filter with a form factor SF close to one (cf Fig. 2-10), several coupled resonant circuits must therefore be used (cf Fig. 2-11).



$$SF \equiv \frac{f_4 - f_3}{f_2 - f_1}$$

Fig 2-10: *Definitions related to filter response.*

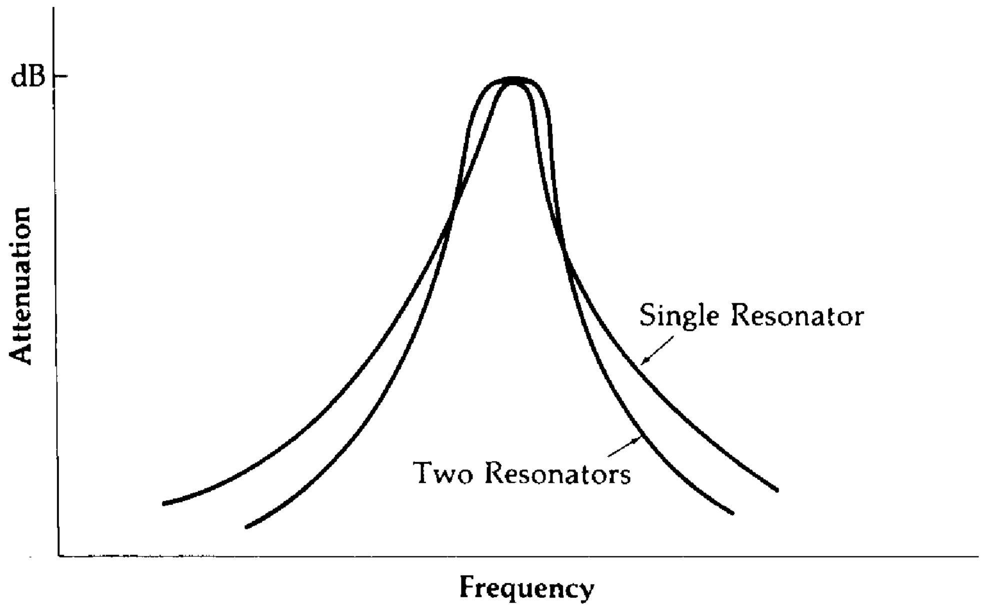


Fig 2-11: *Selectivity with one or two coupled resonant circuits.*

CRITICAL COUPLING

The intensity of the coupling between two resonant circuits will strongly influence the transfer function of the resulting filter. We distinguish three situations:

a)Undercoupling: too little coupling leads to high insertion losses (cf Fig. 2-10 and 2-12);

b)Overcoupling: too much coupling will misalign the two resonant circuits, which results in peaks at the extremities of the passband and a dip in the passband;

c)Critical coupling: the two circuits are just coupled enough to avoid insertion losses while keeping a sufficient passband.

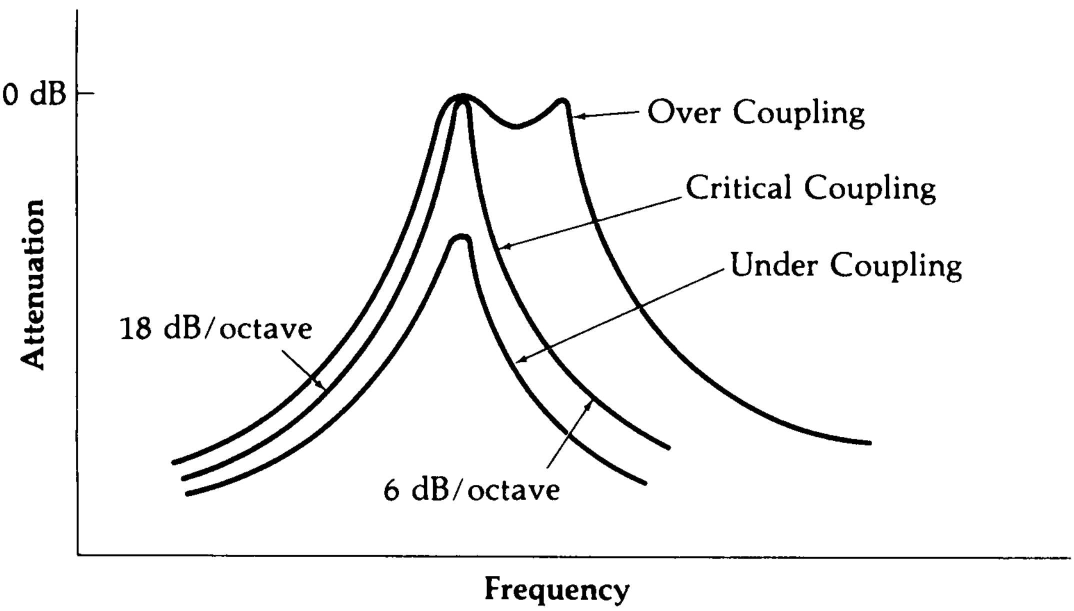


Fig 2-12: *The different coupling situations.*

There are three ways to couple two resonant circuits:

1)Capacitive coupling;

2)Inductive coupling;

3)Coupling by transformer.

Another more sophisticated way to couple resonant circuits is that of LC ladder filters.

CAPACITIVE COUPLING (1/2)

The circuit corresponding to the capacitive coupling of two resonant circuits is presented in Fig. 2-13. The coupling is realized by the capacitor C_c whose value determines whether the coupling is under, at, or over the critical level.

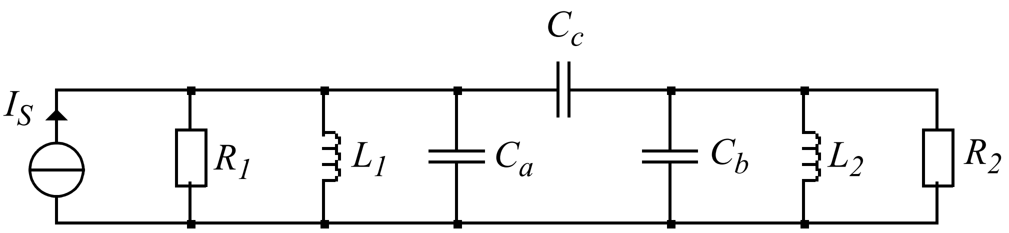


Fig 2-13: *Capacitive coupling of two resonant circuits.*

The circuit in Fig. 2-13 can be modified by considering that the coupling admittance (of C_c) can be replaced by the equivalent circuit of Fig. 2-14.

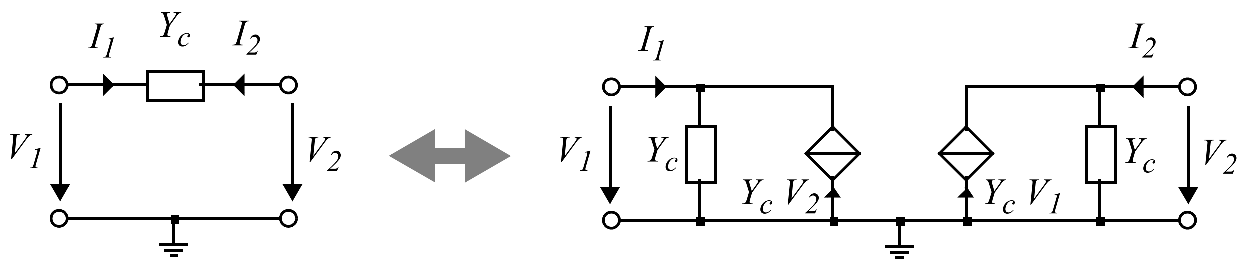


Fig 2-14: *Equivalent circuit of the coupling admittance.*

Applying this substitution to the circuit of Fig. 2-13 we obtain the circuit in Fig. 2-15 in which we clearly see two parallel resonant circuits with admittance Y_1 and Y_2 , along with the coupling by the dependent current sources.

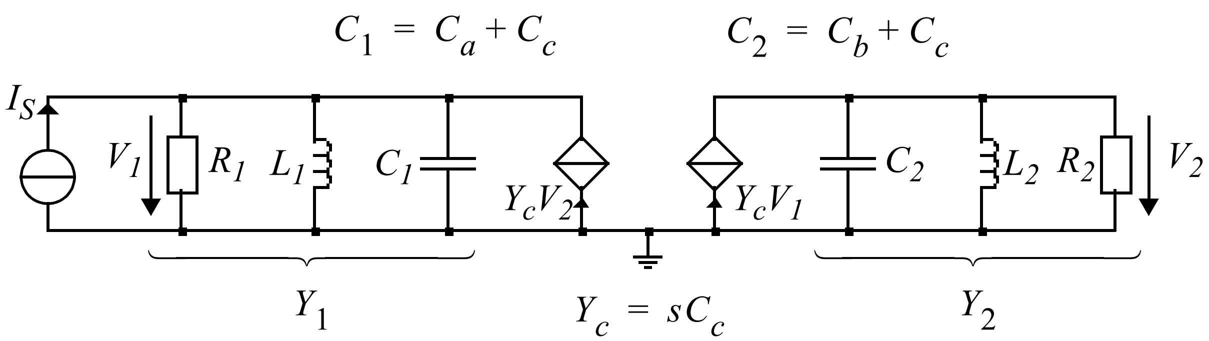


Fig 2-15: *Equivalent circuit for circuit in Fig. 2-13.*

CAPACITIVE COUPLING (2/2)

Each resonant circuit in Fig. 2-15 is characterized by its resonant frequency and its quality factor:

$$\omega_{0i} \equiv 1/\sqrt{L_i C_i} \qquad Q_i \equiv R_i/(\omega_{0i} L_i) = \omega_{0i} R_i C_i \qquad i = 1, 2 \qquad (2.26)$$

The admittance of each of the resonant circuits is given by:

$$Y_i = \frac{1}{R_i} \cdot \left[1 + jQ_i\left(\frac{\omega}{\omega_{0i}} - \frac{\omega_{0i}}{\omega}\right)\right] = \frac{1}{R_i} \cdot [1 + jQ_i x_i] \qquad i = 1, 2 \qquad (2.27)$$

The transimpedance is thus given by:

$$Z_m(j\omega) \equiv \frac{V_2}{I_S} = \frac{Y_c}{Y_1 Y_2 - Y_c^2} \qquad (2.28)$$

Considering that the resonant frequencies are equal, $\omega_{01} = \omega_{02} = \omega_0$, we get:

$$Z_m(j\omega) = \frac{\frac{j\omega}{\omega_0} KR}{1 - \left(\frac{j\omega}{\omega_0} K\right)^2 - (Qx)^2 + j2qQx} \qquad (2.29)$$

$$R \equiv \sqrt{R_1 R_2} \quad Q \equiv \sqrt{Q_1 Q_2} \quad q \equiv \frac{1}{2}\frac{(Q_1 + Q_2)}{\sqrt{Q_1 Q_2}} \quad k \equiv \frac{C_c}{\sqrt{C_1 C_2}} \quad K \equiv kQ = \omega_0 R C_c$$

Q is the *average quality factor*, and q the *quality factor disparity coefficient* which is minimum and equal to one for $Q_1 = Q_2$. It is represented in Fig. 2-16 as a function of the ratio Q_2/Q_1. k is the *coupling coefficient* which is always less than one because C_1 and C_2 are both greater than C_c (cf Fig. 2-15).

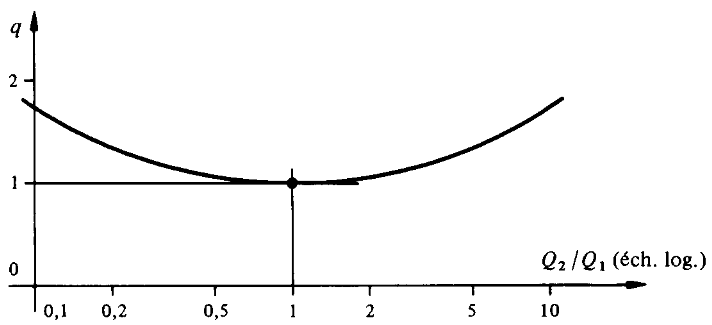
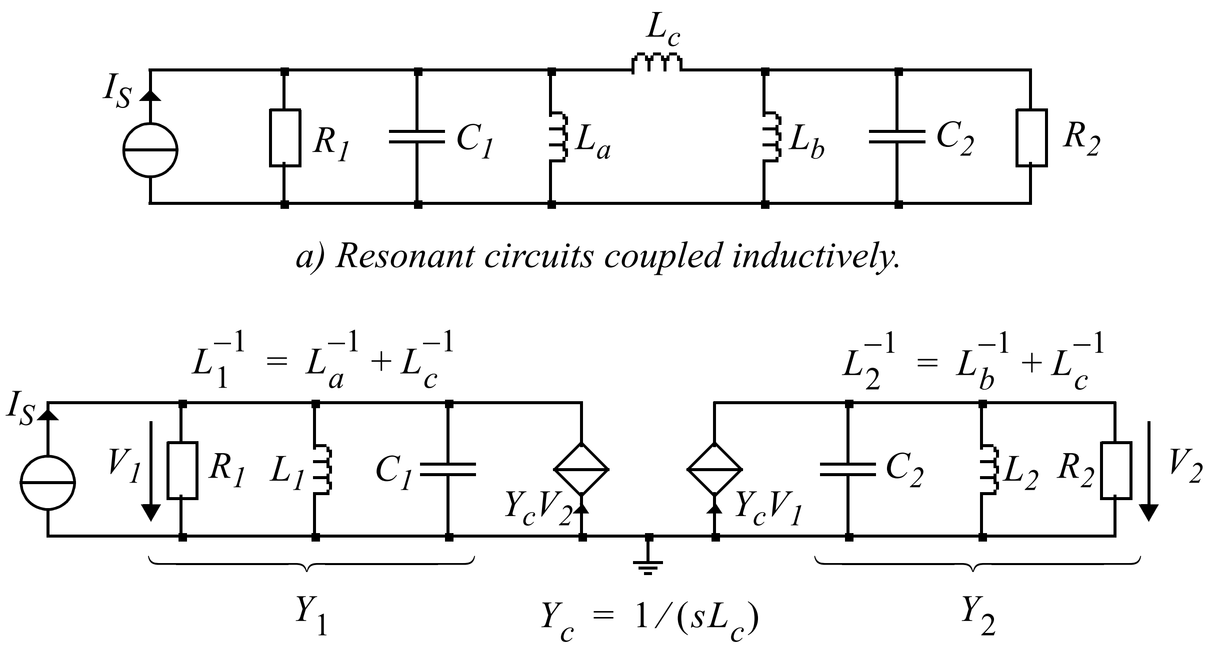


Fig 2-16: *Quality factor disparity coefficient.*

INDUCTIVE COUPLING

The coupling between two parallel resonant circuits can also be carried out by using a coupling inductor L_c, as shown in Fig. 2-17 a).



a) Resonant circuits coupled inductively.

b) Equivalent circuit for a)

Fig 2-17: *Resonant circuits coupled inductively.*

Eqn. 2.28 remains true with $Y_c = 1/(sL_c)$. In the case in which the resonant frequencies are equal, we get:

$$Z_m(j\omega) = \frac{\frac{\omega_0}{j\omega}KR}{1 - \left(\frac{\omega_0}{j\omega}K\right)^2 - (Qx)^2 + j2qQx} \tag{2.30}$$

All the magnitudes defined in (2.29) remain valid except K which becomes $K \equiv kQ = R/(\omega_0 L_c)$ where k is given by:

$$k = \frac{\sqrt{L_1 L_2}}{L_c} \tag{2.31}$$

COUPLING BY TRANSFORMER

The coupling of two parallel resonant circuits can also be carried out by using a transformer as shown in Fig. 2-18.

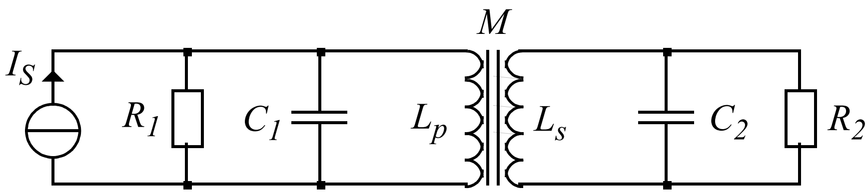


Fig 2-18: *Resonant circuits coupled by a transformer.*

The circuit of Fig. 2-18 can be modified by using the equivalence relation shown in Fig. 2-19. The circuit in Fig. 2-18 is thus returned to that of Fig. 2-17.

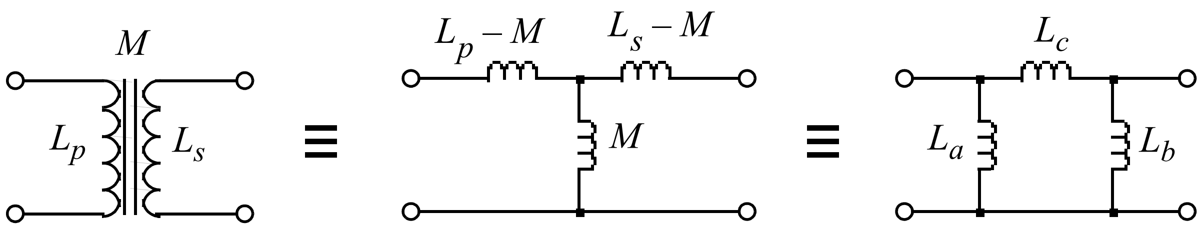


Fig 2-19: *Equivalent circuit of a coupling transformer.*

The relationships obtained for the circuit with inductive coupling are therefore also valid for coupling by transformer, by using the following equalities:

$$L_1 \equiv L_a // L_c = L_p(1 - k^2)$$

$$L_2 \equiv L_b // L_c = L_s(1 - k^2)$$

$$L_c = \frac{\sqrt{L_p L_s}}{k}(1 - k^2) = \frac{\sqrt{L_1 L_2}}{k} \qquad (2.32)$$

$$k = \frac{M}{\sqrt{L_p L_s}} = \frac{\sqrt{L_1 L_2}}{L_c}$$

CRITICAL AND TRANSITIONAL COUPLING (1/2)

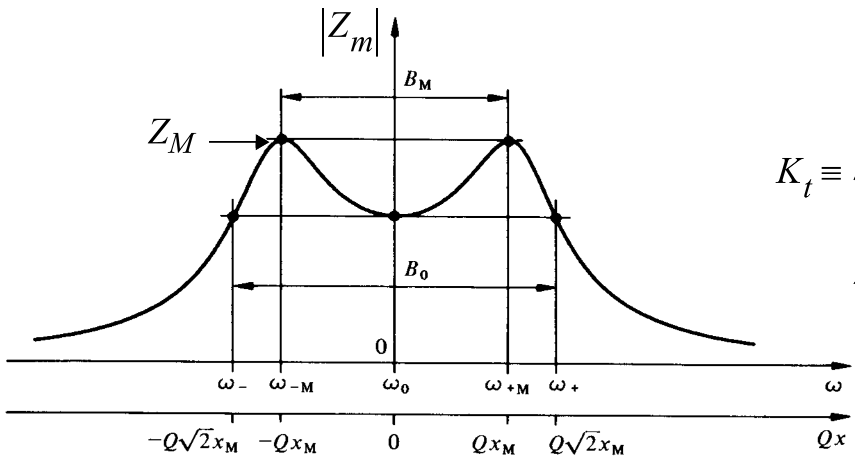
The value of the transimpedance Z_m at resonance is obtained by setting $\omega = \omega_0$ in equations (2.29) and (2.30):

$$Z_m(\omega_0) = \pm j\frac{KR}{1+K^2} = \pm jZ_{max}\frac{2K}{1+K^2} \qquad (2.33)$$

The + sign corresponds to capacitive coupling and the - sign to inductive (or transformer). The magnitude of the transimpedance at resonance $|Z_m|$ depends on the coupling factor K. Critical coupling results for the value of K which maximizes $|Z_m|$, which is $K = 1$:

$$Z_{max} \equiv |Z_m|\big|_{K=1} = R/2 \qquad (2.34)$$

In fact, it corresponds to the maximum transfer of power to the resistor R_2 and therefore to impedance matching.
When the coupling factor K is higher than a certain value called the transitional coupling K_t, the magnitude of the transimpedance shows two peaks at the frequencies ω_{-M} and ω_{+M} as indicated in Fig. 2-20.



$$K > K_t$$

$$K_t \equiv \sqrt{2q^2 - 1} = \sqrt{\frac{1}{2}\left(\frac{Q_1}{Q_2} + \frac{Q_2}{Q_1}\right)}$$

$$Z_M \equiv \frac{KR}{2q\sqrt{1+K^2-q^2}}$$

Fig 2-20: *Peaks due to overcoupling.*

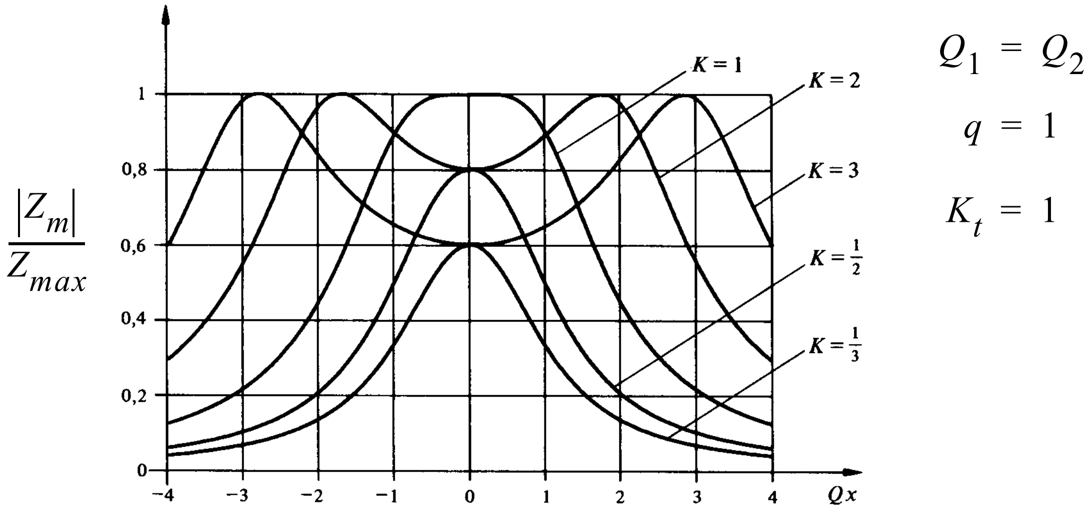
The value of the maxima Z_m is given by:

$$Z_M = \frac{KR}{2q\sqrt{1+K^2-q^2}} \qquad (2.35)$$

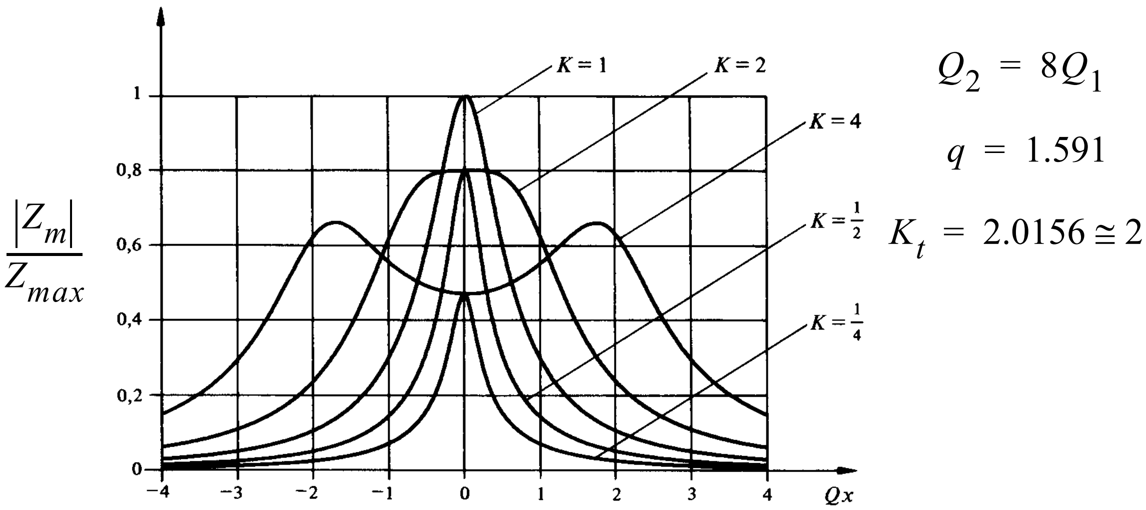
One remarks that $Z_M = Z_{max}$ for $q = 1$ or for $Q_1 = Q_2$.

CRITICAL AND TRANSITIONAL COUPLING (2/2)

Fig. 2-21 a) shows the magnitude of Z_m/Z_{max} as a function of misalignment Qx for different values of the normalized coupling coefficient K for the case in which $Q_1 = Q_2$. We remark that for $K > K_t = 1$, the curves show maxima where Z_M equals Z_{max}. Fig. 2-21 b) shows the magnitude of Z_m/Z_{max} for the case in which the quality factors are different. We remark that the value Z_M of the maxima decreases as K increases. According to Eqn. 2.35, it tends toward $R/(2q)$ as $K \rightarrow \infty$.



a) $Q_1 = Q_2$



b) $Q_2 = 8Q_1$

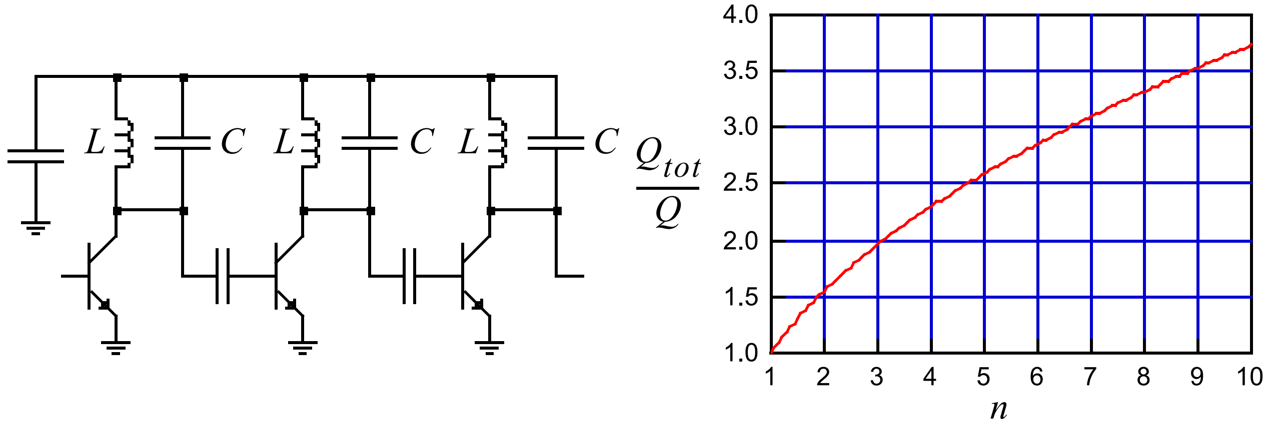
Fig 2-21: *Magnitude of the transimpedance as a function of Qx*

ACTIVE COUPLING

It is also possible to couple resonant circuits using active devices such as transistors. An example of such a coupling is shown in Fig. 2-22 a). In the case in which all the resonant circuits in the diagram in Fig. 2-22 a) are identical (same frequency and quality factor), the global quality factor Q_{tot} of the circuit is approximately given by:

$$Q_{tot} = \frac{Q}{\sqrt{2^{1/n} - 1}} \tag{2.36}$$

where Q is the quality factor of each of the resonant circuits and n the number of resonant circuits.



a) Example.

b) Global quality factor as a function of the number of resonant circuits (Eqn. 2.36).

Fig 2-22: *Active coupling of resonant circuits.*

Fig. 2-22 b) shows the global quality factor normalized to the quality factor of one single resonant circuit as a function of the number of stages. Notice that Q_{tot} increases relatively slowly and therefore that the gain in selectivity is only interesting for a reduced number of resonant stages (typically 3 or 4).